On the topological entropy of discontinuous functions. Strong entropy points and Zahorski classes

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Abstract. The basis of our considerations are the issues dealing with entropy of discontinuous functions (among others almost continuous functions and functions belonging to a fixed Zahorski class). Particular emphasis is on the local aspects of entropy including problems regarding strong entropy points (also almost fixed points).

Keywords: entropy, Darboux function, Zahorski classes, almost fixed point, strong entropy point, almost continuity, Γ-approximation, f-bundle, manifold.

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1. Introduction – historical outline

The theory of (discrete) dynamical systems is extensive and strongly expanding field of mathematics that uses a lot of facts from various branches of mathematics including real analysis. It is interesting, therefore, to consider the issues associating the discrete dynamical systems theory and real analysis. In this chapter we will present some issues concerning entropy of functions belonging to different Zahorski classes.

The results presented in this part are mainly based on the paper [22] and [23]. If we give statements from other publications related to this topic, this will be marked by giving references to the relevant article.

We start with a short historical overview of the problems presented in this chapter.

First, we present some intuitive description of problems connected with information system and information flow. We will not present the issues of information systems in details (basic facts on this subject can be found, among others in [24, 25]). However, assume that we have a set X of elements (information) divided into a finite number of disjoint subsets $\{A_1, A_2, \dots, A_k\}$, which are distinguished on the basis of fixed

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attributes (this partition is denoted by P). Suppose also that we have a probability measure[1] μ on X, so $\sum_{i=1}^{k} \mu(A_i) = 1$. Then we may (see [26]) assign to the partition P the number (the entropy of partition) defined in the following way:

$$H(P) := -\sum_{i=1}^{k} \mu(A_i) \cdot \log \mu(A_i).$$

Roughly speaking, if partition P describes a state of information flow, the number $H(P)$ may be regarded as a "measure of uncertainty". If $H(P) = 0$, then situation is defined precisely – measure is focused on some set A_{i_0} from the partition P (i.e. $\mu(A_{i_0}) = 1$). Moreover, we can say that the higher the entropy of partition is, the greater uncertainty is (in this case, the measure is more evenly distributed over the different sets of the partition).

After a given period of time, elements of X change the values of their attributes and thereby they "move" to the other sets. Perhaps a new partition of X (onto sets measurable with respect to μ) is created. These changes are described by a certain function – let us denote it by ϕ. After the next unit of time, the elements "move" again and we obtain a new partition of X. The changes are described by the function ϕ. It means that in comparison to initial state these changes are described by the function $\phi^2 = \phi \circ \phi$. Going further in this way, we obtain the dynamics of the function ϕ. The entropy of this function (the definition one can find at the end of this section) determines the level of uncertainty of dynamics of function ϕ. If it is 0, then we can talk about a certain stability of this dynamics. If it is greater than 0, we can say that this dynamics is chaotic and the number qualified as the entropy can be considered as a certain kind of "measure of chaos".

In the sixties of the twentieth century R.L. Adler, A.G. Konheim and M.H. McAndrew [1] introduced the notion of the topological entropy of a continuous function $f : X \circlearrowleft$ (i.e $f : X \to X$) defined on a compact space X. In 1971 T. Goodman [11] proved the variational principle determining the relationship between the topological entropy and the entropy with respect to measure (cf. Theorem 7.7). Earlier, in 1969, L.W. Goodwyn [12] proved that for a fixed invariant measure, the entropy of function with respect to this measure is not greater than its topological entropy.

In the case of functions important from the point of view of the real analysis theory (e.g. functions belonging to a fixed Zahorski class) their entropy with respect to a measure is difficult to use because of the necessity of constructing and selecting invariant measures. Since each Zahorski class contains also discontinuous functions, we can not directly apply the original definition. Fortunately, in [7] it has been shown that there is a possibility of using existing definitions in the case of discontinuous functions. It is worth noting that until now the topological entropy has been mainly related to Darboux-like functions ([7, 22, 23, 21, 17, 14]).

Now, we shortly recall definition of a topological entropy given for continuous function by R. Bowen [3] and E. Dinaburg [8] and extended to an arbitrary function by Čiklová [7].

[1] Assumptions concerning this measure and the corresponding functions are described in details in Section 7.

Let (X,ρ) be a compact metric space, $f : X \circlearrowleft$ be a function, $\varepsilon > 0$ and $n \in \mathbb{N}$. A set $M \subset X$ is (n,ε)-*separated* if for each $x,y \in M$, $x \neq y$ there is $0 \leqslant i < n$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$. Let

$$s_n(\varepsilon) = \max\{\operatorname{card}(M) : M \subset X \text{ is } (n,\varepsilon)\text{-separated set}\}.$$

The *topological entropy of the function* f is the number

$$h(f) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \left[\frac{1}{n}\log\left(s_n(\varepsilon)\right)\right].$$

A detailed analysis of behaviour of various functions pointed out the desirability of searching subsets of a domain of a function, or even points, on which the behaviours of a function having significant impact on the value of its topological entropy, are focused. Description of such analysis will be presented in sections 3 and 4. This analysis led us to distinguish an important object – "a strong entropy point of a function" (the respective definition is presented in Section 4). Of course, not every function (even if we assume its continuity) has a strong entropy point. For this reason, issues related to approximating functions by other functions with a strong entropy point seemed to be interesting ([22, 23, 14]). At the same time it is interesting to approximate a function by functions with "a lot of continuity points". Therefore, uniform convergence is not a very convenient tool. Moreover, it is worth noting that if f belongs to any Zahorski class, then f is almost continuous i.e each open set containing the graph of this function contains a graph of a continuous function (see Proposition 6.1). For these reasons, it is quite natural to consider Γ-approximation. More information about this kind of approximation is presented in Section 5.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. We use the letter λ to denote the Lebesgue measure in $\mathbb{R}$.

The symbol $\mathfrak{D}(X,Y)$ ($\mathfrak{B}_1(X,Y)$) denotes the family of all functions $f : X \to Y$ which are Darboux functions (of first Baire class). In both the above notations if $X = Y$ we will write only one X, e.g. $\mathfrak{D}(X)$ instead of $\mathfrak{D}(X,X)$. Moreover, if we wish to consider the intersection of two classes of functions, we shall write them next to each other (e.g. $\mathfrak{D}\mathfrak{B}_1(X,Y)$ or $\mathfrak{D}\mathfrak{B}_1(X)$). Furthermore, we write $\mathfrak{D}$ and $\mathfrak{B}_1$ if $X = Y = \mathbb{R}$.

Let $f : X \circlearrowleft$. Then we define $f^0(x) = x$ and $f^i(x) = f(f^{i-1}(x))$ for any $i \in \mathbb{N}$. If $A \subset X$ and $n \in \mathbb{N}$ then $f^{-n}(A) = \{x \in X : f^n(x) \in A\}$. A point $x_0 \in X$ is a *fixed point* of a function f if $f(x_0) = x_0$. A set of all fixed points of f we denote by $\operatorname{Fix}(f)$. We say that a space X has the *fixed point property* if $\operatorname{Fix}(f) \neq \emptyset$ for any continuous function $f : X \circlearrowleft$. If $x_0 \in X$, $f^m(x_0) = x_0$ for some $m \in \mathbb{N}$ and $f^n(x_0) \neq x_0$ for any $n \in \{1,\dots,m-1\}$, then we call a point x_0 a *periodic point* of f of *prime period* m. The symbol $\operatorname{Per}_m(f)$ stands for a set of all periodic points of f of prime period m. W say that $x_0 \in X$ is a *periodic point of* f if $x_0 \in \bigcup_{n\in\mathbb{N}} \operatorname{Per}_n(f)$. For any $x \in X$ the *orbit of f at the point x* is the set $\{x, f(x), f^2(x), f^3(x), \dots\}$. If $x \in \mathrm{Per}_m(f)$ for some $m \in \mathbb{N}$ then the orbit of f at the point x is called a *periodic orbit of f of period m*.

We say that functions $f, g : X \circlearrowleft$ are *conjugate via a homeomorphism* $\phi : X \circlearrowleft$ if $\phi \circ f = g \circ \phi$.

Let $f : X \to Y$, $A \subset X$ and $B \subset Y$. We say that a set A *f-covers* a set B (denoted by $A \underset{f}{\to} B$) if $B \subset f(A)$. Moreover, the restriction of f to the set A is denoted by $f \restriction A$. The symbol $\mathrm{card}(A)$ stands for cardinality of A.

Let (X, ρ) be a metric space. The symbol $\mathrm{dist}(x, A)$, where $x \in X$ and $A \subset X$, stands for a distance from the point x to the set A. If $x_0 \in X$ and $r > 0$, then we use the symbol $B(x_0, r)$ to denote an open ball with the centre at x_0 and the radius r.

We say that a topological space X is an *m-dimensional topological manifold with boundary* if X is a second countable Hausdorff space and every point $q \in X$ has a neighborhood that is homeomorphic to the m-dimensional upper half space $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geqslant 0\}$ (see [15]).

We say that $\alpha \in \mathbb{R}$ is a *left* (*right*) *range* of $f : \mathbb{R} \circlearrowleft$ at $x_0 \in \mathbb{R}$ if $f^{-1}(\alpha) \cap (x_0 - \delta, x_0) \neq \emptyset$ ($f^{-1}(\alpha) \cap (x_0, x_0 + \delta) \neq \emptyset$) for any $\delta > 0$. The symbols $R^+(f, x_0)$ and $R^-(f, x_0)$ stand for the sets of all left and all right ranges of f at x_0, respectively.

Let $f : [0,1] \circlearrowleft$ be a Darboux function. A point $x_0 \in (0,1)$ ($x_0 = 0$, $x_0 = 1$) is an *almost fixed point* of f (for short $x_0 \in \mathrm{Fix}_a(f)$) iff $x_0 \in \mathrm{int}(R^-(f, x_0)) \cup \mathrm{int}(R^+(f, x_0))$ ($x_0 \in \mathrm{int}(R^+(f, x_0))$, $x_0 \in \mathrm{int}(R^-(f, x_0))$).

For any finite family $\{I_1, \dots I_n\}$ of closed intervals contained in $[0,1]$ we define a matrix $M_f(I_1, \dots I_n) = [a_{ik}]_{i,k \leqslant n}$ in the following way: $a_{ik} = 1$ if $I_i \underset{f}{\to} I_k$ and $a_{ik} = 0$ otherwise. A maximal absolute value of an eigenvalue of this matrix will be denoted by $\sigma(M_f(I_1, \dots I_n))$.

3. Bundles connected with function

In this section we will concentrate on a metric space (X, ρ). We write it X for short.

We start with the example of the function (having some special property) defined on the square $I^2 = [0,1] \times [0,1]$ with the Euclidian metric.

In the square I^2 we consider two rectangles P_1 with vertices a_1, a_4, b_4, b_1 and P_2 with vertices c, d, e, k. We divide P_1 and P_2 into rectangles as it is shown in Figure 1. We define the function $f : I^2 \circlearrowleft$ in the following way: $f((0,0)) = (0,0)$ and $f(t) = t$ if t belongs to the rectangle P with vertices x, q, r, b_4. Moreover, we define the function f on P_1 as follows: we "stretch" each rectangle from the partition of P_1 onto the rectangle P_1 in such a way that the line segment with endpoints a_3 and b_3 is converted to the line segment with endpoints a_1 and b_1, the line segment with endpoints a_2 and b_2 is converted to the line segment with endpoints a_4 and b_4. On the segments $a_1 b_1$ and $a_4 b_4$ function f is the identity. Analogously we define the function f on the rectangle P_2. Next we extend f to continuous function defined on I^2.

It is easy to see that $h(f) > 0$. However, one can show that $h(f \restriction P) = 0$ and $h(f \restriction P_2) > h(f \restriction P_1)$.

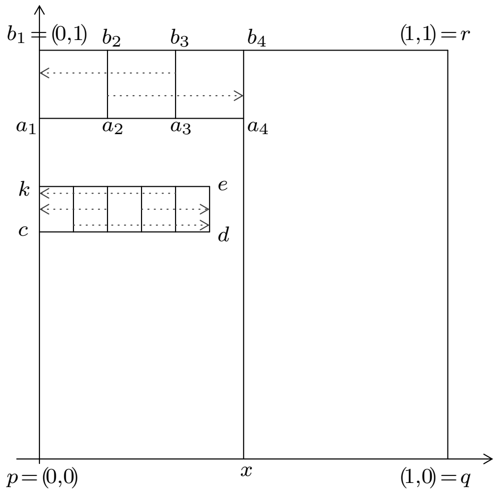

Fig. 1

The natural question arises: Is it possible to identify subsets of domain which have a particular impact on the value of the entropy of the whole function? This question is interesting because, as the next example will show, sometimes even small changes have a significant impact on the value of the entropy of a function. This time the example will regard discontinuous functions, however it is not difficult to show analogous example for a continuous function.

So, let us consider two functions $g, k : [0,1] \circlearrowleft$ whose graphs are presented in Figure 2.

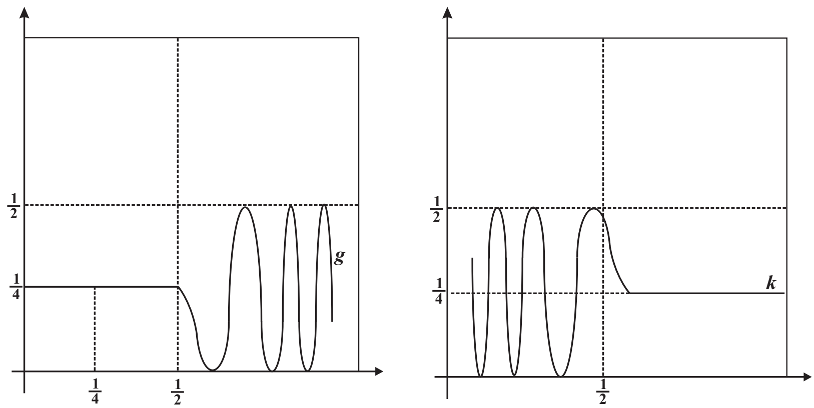

Fig. 2

We see that $g(0) = k(1) = \frac{1}{4}$ and the graphs of g and k are more and more "dense" near the point 1 and 0, respectively (their graphs are of the same type as that of $\sin(\frac{1}{x})$). Notice that "behaviours" of these functions are similar to each other, but specificity of their behaviours is different on different sets. As a result, $h(g) = 0$, whereas $h(k) = +\infty$. In that sense, the function g may be regarded as "predictable" and the function k as "strongly chaotic".

Similar considerations are presented in [23], where one can find other examples of functions slightly differing from each other, but the entropy of one of them is 0 and of the second one is greater then 0.

Simultaneously, in consideration of the complexity of entropy definition, one can ask another question: Is there a simple way to indicate the sets which have a decisive impact on the value of entropy of a fixed function?

The next part of the chapter is devoted to the answer to the above question. Some basis of these considerations one can find in [2, 18, 20]. Following [23], we will consider the concept of f-bundle, which is some generalization of the notion horseshoe (e.g. [2]).

Let $f : X \circlearrowleft$. A pair $(\mathcal{F}, J) = B_f$, where $\mathcal{F}$ is a family of pairwise disjoint (nonsingletons) continuums in X and $J \subset X$ is a connected set such that $A \underset{f}{\to} J$ for any $A \in \mathcal{F}$ is called an *f-bundle.* Moreover, if we additionally assume that $A \subset J$ for all $A \in \mathcal{F}$ then such an f-bundle is called an *f-bundle with dominating fibre.* By the cardinality of B_f (denoted by $\mathrm{card}(B_f)$) we will mean the cardinality of the family $\mathcal{F}$.

Let $f : X \circlearrowleft$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $B_f = (\mathcal{F}, J)$ be an f-bundle. A set $M \subset \bigcup \mathcal{F}$ is *(B_f, n, ε)-separated* if for each $x, y \in M$, $x \neq y$ there is $0 \leqslant i < n$ such that $f^i(x), f^i(y) \in J$ and $\rho(f^i(x), f^i(y)) > \varepsilon$. If

$$s_n^{B_f}(\varepsilon) = \max\{\mathrm{card}(M) : M \subset X \text{ is } (B_f, n, \varepsilon)\text{-separated set}\},$$

then the *entropy of the f-bundle B_f* is defined in the following way:

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log \left(s_n^{B_f}(\varepsilon) \right) \right].$$

Theorem 3.1 ([23])**.** *Let $f : X \circlearrowleft$ be an arbitrary function and $B_f = (\mathcal{F}, J)$ be an f-bundle with dominating fibre. Then $h(B_f) \geq \log(\mathrm{card}(B_f))$ if B_f is finite and $h(B_f) = +\infty$ otherwise.*

Recall again the example of the function $f : I^2 \circlearrowleft$ presented before.

From now on, the symbol $P(t_1, t_2, t_3, t_4)$ will stand for a rectangle with vertices t_1, t_2, t_3, t_4. Then, for the function f, one can consider the following f-bundles with dominating fibres: $(\mathcal{F}_i, J_i)$ $(i = \{1, 2, 3\})$: $\mathcal{F}_1 = \{P(x, q, r, b_4)\}$ and $J_1 = P(x, q, r, b_4)$, $\mathcal{F}_2 = \{P(a_3, a_4, b_4, b_3), P(a_1, a_2, b_2, b_1)\}$ and $J_2 = P(a_1, a_4, b_4, b_1)$; $\mathcal{F}_3$ consists of the grey rectangles (see Figure 3) and $J_3 = P(c, d, e, k)$.

Theorem 3.1 implies[2] $h((\mathcal{F}_1, J_1)) \geqslant 0$, $h((\mathcal{F}_2, J_2)) \geqslant \log 2$ and $h((\mathcal{F}_3, J_3)) \geqslant \log 3$.

[2] In considered rectangles $P(a_1, a_4, b_4, b_1)$ and $P(c, d, e, k)$ one can find f-bundles of cardinality greater then those presented. However it would require more complicated notation unnecessary for our further considerations.

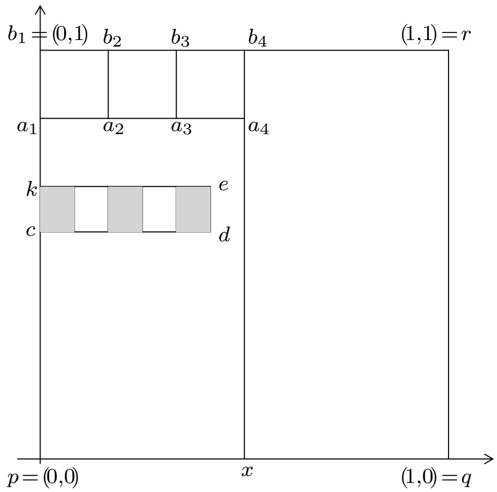

Fig. 3

We say that a map f is *chaotic in the sense of Li and Yorke* if f has orbits of arbitrarily large periods and there exists an uncountable set B (called a *scrambled set*) such that for every $x, y \in B$ such that $x \neq y$ and every periodic point z we have

1. $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$;
2. $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$;
3. $\limsup_{n\to\infty} |f^n(x) - f^n(z)| > 0$.

Theorem 3.2 ([23]). *Let $f : [0,1] \circlearrowleft$ be a continuous function. If there exists an f-bundle $B_f = (\mathcal{F}, J)$ with dominating fibre such that* $\mathrm{card}(B_f) \geqslant 2$*, then f is chaotic in the sense of Li and Yorke.*

4. Strong entropy points

In this section X still stands for a metric space (X, ρ).

Let us modify in the following way the function $f : I^2 \circlearrowleft$ considered previously. At first, we have considered two rectangles $P(a_1, a_4, b_4, b_1)$ and $P(c, d, e, k)$, their partitions onto the smaller rectangles and the function "stretching" each of the rectangles from partitions onto the whole initial rectangle. In this way, the f-bundles $(\mathcal{F}_2, J_2)$ and $(\mathcal{F}_3, J_3)$ described above, have been created. Assume that we consider next rectangles whose lengths of sides converge to zero, partitions onto more and more parts and function f "stretching" each of the rectangles from partitions onto the whole initial rectangle. In this way we can create a sequence of f-bundles (see Figure 4).

It is easy to notice that the entropy of each of successive bundles will be greater and greater. The observation with the above example may be an intuitive illustration of the following definitions.

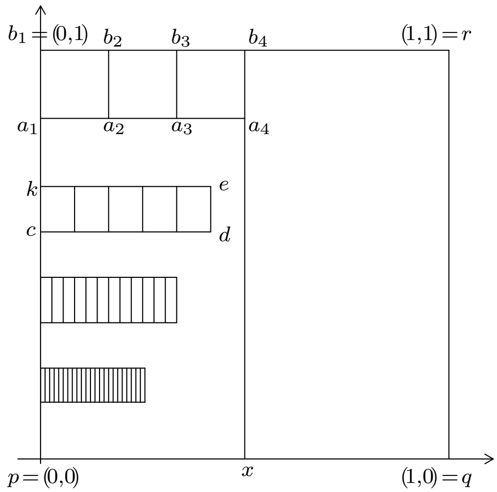

Fig. 4

Let $f : X \circlearrowleft$. We shall say that a sequence of f-bundles $B_f^k = (\mathcal{F}_k, J_k)$ *converges to a point* x_0 ($B_f^k \underset{k\to\infty}{\longrightarrow} x_0$), if for any $\varepsilon > 0$ there exists $k_0 \in N$ such that $\bigcup \mathcal{F}_k \subset B(x_0, \varepsilon)$ and $B(f(x_0), \varepsilon) \cap J_k \neq \emptyset$ for any $k \geqslant k_0$. Having the above notion we can define multifunction $E_f : X \multimap \mathbb{R} \cup \{+\infty\}$ in the following way:

$$E_f(x) = \{\limsup_{n\to\infty} h(B_f^n) : B_f^n \underset{n\to\infty}{\longrightarrow} x\}.$$

Obviously for an arbitrary function $f : X \circlearrowleft$ and a point $x_0 \in X$ we have $E_f(x_0) \neq \emptyset$.

An important issue in considering multivalued functions is, in some sense, "regularity" of their values. The most essential kind of such "regularity" deals with closedness. The next theorems regard simple observations connected with this issue.

Theorem 4.1 ([23])**.** *If $f : X \circlearrowleft$ is an arbitrary function, then $E_f(x)$ is a closed set for any $x \in X$.*

Theorem 4.2 ([23])**.** *If $f : X \circlearrowleft$ is a continuous function, then E_f is a closed multifunction.*

Return again to the function $f : I^2 \circlearrowleft$ described at the begining of this section. In each case of considered f-bundles, their entropy is finite, but according to Theorem 3.1 $h(f) = +\infty$. Notice that in this case the entropy of the function is focused around the point $(0, 0)$. Thus let us adopt the definition which may mean the entropy of a function at a point.

For any function $f : X \circlearrowleft$ and $x \in X$ the *entropy of f at point x*, denoted by $e_f(x)$, is defined to be $\sup E_f(x)$.

Theorem 4.3 ([23])**.** *If $f : X \circlearrowleft$ is an arbitrary function and $x \in X$, then $e_f(x) \leqslant h(f)$.*

Theorem 4.4 ([23])**.** *If $f : X \circlearrowleft$ is a continuous function, then the function $e_f(x)$ is an upper semicontinuous selection.*

In this section we focus our attention also on strong entropy points. We shall say that a point $x_0 \in X$ is a *strong entropy point* of a function $f : X \circlearrowleft$ if $h(f) \in E_f(x_0)$ and $x_0 \in \mathrm{Fix}(f)$. The family of all strong entropy points of a function f will be denoted by $\mathfrak{E}_s(f)$.

First we note the obvious statements.

Theorem 4.5 ([23])**.** *A point $x_0 \in X$ is a strong entropy point of a function $f : X \circlearrowleft$ if and only if $e_f(x_0) = h(f)$ and $x_0 \in \mathrm{Fix}(f)$.*

The above observations imply immediately the following statement.

Theorem 4.6. *Let $f : X \circlearrowleft$ be an arbitrary function and $x_0 \in X$.*

(a) If $x_0 \in \mathrm{Fix}(f)$ and $e_f(x_0) = +\infty$, then $x_0 \in \mathfrak{E}_s(f)$.
(b) If $x_0 \in \mathrm{Fix}(f)$ and $+\infty \in E_f(x_0)$, then $x_0 \in \mathfrak{E}_s(f)$.

Notice that in our example of the function $f : I^2 \circlearrowleft$, $e_f((0,0)) = +\infty$, so the entropy of f at the point $(0,0)$ coincides with the entropy of the whole function and what is more, the point $(0,0)$ is a fixed point of f. It means that $(0,0)$ is a strong entropy point of f.

The function $f : I^2 \circlearrowleft$ considered above is continuous. The natural question arises whether the similar considerations may be referred to discontinuous functions. The answer will be contained in Sections 5 and 9.

We will finish this part of the paper with the theorem showing that the notion of strong entropy point is interesting from the point of view of dynamical systems.

Theorem 4.7. *Let functions $f : X \circlearrowleft$ and $g : X \circlearrowleft$ be conjugate. Then $\mathfrak{E}_s(f) \neq \emptyset$ if and only if $\mathfrak{E}_s(g) \neq \emptyset$.*

5. Almost continuous functions defined on m-dimensional manifold

In this section we will consider approximations of some functions by functions having strong entropy point. From the point of view of known facts regarding discrete dynamical systems and topology, the issue of approximation by use of continuous functions is very importatnt. On the other hand, we aim to combine these considerations with real functions theory regarding, among others, Zahorski classes. Approximation of discontinuous functions by continuous functions eliminates the possibility of considering uniform approximation (i.e. approximation by use of topology of uniform convergence). Simultaneously, a definition of strong entropy points indicates that our considerations should be directed towards classes of functions having a fixed point. Taking into account all the above facts, it is appropriate to consider almost continuous functions and to investigate graph-approximation (Γ-approximation).

Let (X, τ) be a topological space and $\mathcal{K}$ be some class of functions from X into itself. We shall say that a function $f : X \circlearrowleft$ is Γ-approximated by functions belonging to $\mathcal{K}$ if for each open set $U \subset X \times X$ containing the graph of f, there exists $g \in \mathcal{K}$ such that the graph of g is a subset of U.

In this section we will focus on almost continuous functions in the sense of Stallings [27], namely on functions $f : X \circlearrowleft$ which are Γ-approximated by continuous functions. It is known that the family of all functions $f : [0,1] \circlearrowleft$ which are almost continuous and of first Baire class is equal to the family $\mathfrak{D}\mathfrak{B}_1([0,1])$ (see [4]). However, there exist topological spaces (X, τ_X), (Y, τ_Y) and an almost continuous function $f : X \to Y$ such that $f \in \mathfrak{B}_1(X, Y)$ and $f \notin \mathfrak{D}(X, Y)$ (see [18]). Moreover, we have the following fact.

Theorem 5.1 ([27]). *If X is a Hausdorff space with the fixed point property then each almost continuous function $f : X \circlearrowleft$ has a fixed point.*

The above theorem plays an important role in the proof of the following claim.

Theorem 5.2 ([23]). *Let X be a compact, m-dimensional manifold with boundary having the fixed point property and $f : X \circlearrowleft$ be a function. The following conditions are equivalent:*

(1) The function f is almost continuous.
(2) The function f can be Γ-approximated by continuous functions having a strong entropy point.
(3) The function f can be Γ-approximated by continuous functions having infinite topological entropy.
(4) The function f can be Γ-approximated by discontinuous but almost continuous functions having a strong entropy point.
(5) The function f can be Γ-approximated by discontinuous but almost continuous functions having infinite topological entropy.

6. The Zahorski classes

Working on the issues regarding derivatives, Zygmunt Zahorski distinguished a hierarchy of classes of functions. Following [29], let us start with definitions of some classes of sets. All these classes of sets consist of some subsets of $\mathbb{R}$. The class $\mathfrak{M}_0$ consists of the empty set and all nonempty sets E of type F_σ such that every point of E is a point of bilateral accumulation of E. The family of all nonempty sets E of type F_σ such that every point of E is a point of bilateral condensation of E complemented by the empty set constitutes the class $\mathfrak{M}_1$. A set E belongs to the class $\mathfrak{M}_2$ if it is empty or if it is a nonempty set of type F_σ and for each $x \in E$ and any $\varepsilon > 0$ sets $(x, x+\varepsilon) \cap E$ and $(x-\varepsilon, x) \cap E$ have a positive measure. The class $\mathfrak{M}_3$ consists of all nonempty sets E of type F_σ such that there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed sets such that $E = \bigcup_{n\in\mathbb{N}} K_n$ and a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of numbers such that $0 \leqslant \eta_n < 1$ $(n \in \mathbb{N})$ and for each $n \in \mathbb{N}$, each $x \in K_n$ and each $c > 0$ there exists a number $\varepsilon(x, c) > 0$ such that if h and h_1 satisfy conditions $h \cdot h_1 > 0$, $\frac{h}{h_1} < c$, $|h + h_1| < \varepsilon(x, c)$, then

$$\frac{\lambda(E \cap (x+h, x+h+h_1))}{|h_1|} > \eta_n.$$

In addition, we assume that the empty set belongs to the class $\mathfrak{M}_3$. A slight change in the definition of the class $\mathfrak{M}_3$ leads us to the class $\mathfrak{M}_4$. More specifically, in this case, we replace the above condition $0 \leqslant \eta_n < 1$ $(n \in \mathbb{N})$ with the condition $0 < \eta_n < 1$ $(n \in \mathbb{N})$. We say that E belongs to the class $\mathfrak{M}_5$ if it is empty or if it is a nonempty set of type F_σ and for each $x \in E$ we have

$$\lim_{h \to 0^+} \frac{\lambda(E \cap [x-h, x+h])}{2h} = 1, \tag{1}$$

that is every point of E is a density point of E.

It is worth adding that to check whether a nonempty set E of type F_σ belongs to the class $\mathfrak{M}_3$ it is enough to show the following condition: for each $x \in E$ and each sequence $\{I_n\}_{n \in \mathbb{N}}$ of closed intervals converging to x (i.e. $\lim\limits_{n\to\infty} \operatorname{dist}(x, I_n) = 0$) and not containing x such that $\lambda(I_n \cap E) = 0$ for each $n \in \mathbb{N}$, we have $\lim\limits_{n \to \infty} \frac{\lambda(I_n)}{\operatorname{dist}(x, I_n)} = 0$.

Using the above hierarchy of sets we can define some classes of functions. Let $i \in \{0, 1, \ldots, 5\}$. We say that a function $f : \mathbb{R} \circlearrowleft$ belongs to the class $\mathcal{M}_i$, if sets $\{x : f(x) > \alpha\}$ and $\{x : f(x) < \alpha\}$ belong to the class $\mathfrak{M}_i$ for any $\alpha \in \mathbb{R}$. Certainly one can define similar classes for functions $f : [0,1] \circlearrowleft$.

Moreover, to simplify notation, let the symbol $\mathcal{M}_6$ stand for the family of all continuous functions. It is easy to see that

$$\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5 \supset \mathcal{M}_6.$$

It is well known that all the above inclusions, except the first one from the left, are proper and $\mathcal{M}_0 = \mathcal{M}_1 = \mathfrak{D}\mathfrak{B}_1$ ([29, 5]). Therefore, further instead of writing: the function f is a Darboux function and of first Baire class, we will write briefly: $f \in \mathcal{M}_1$.

In [29] one can also find the property saying that the class $\mathcal{M}_5$ coincides with the family of approximately continuous functions i.e the family of functions f having the following properties (see [9, 10]): for any $x \in \mathbb{R}$ there exists a Lebesgue measurable set E_x such that x is a density point of E_x (see condition (1)) and

$$f(x) = \lim_{\substack{t \to x,\\ t \in E_x}} f(t).$$

Additionally, it is shown there that each derivative (a function f is a derivative if there exists a function g such that $f = g\prime$) belongs to the class $\mathcal{M}_3$ and each bounded derivative is in $\mathcal{M}_4$.

Furthermore, it is known that each bounded function from the class $\mathcal{M}_5$ is a derivative (see [10]).

Moreover, we have the following, commonly known, facts.

Proposition ([29]). *For any set $E \in \mathfrak{M}_4$ there exists a bounded derivative $f : \mathbb{R} \circlearrowleft$ such that $f(x) = 0$ for $x \notin E$ and $f(x) \in (0,1)$ for $x \in E$.*

Proposition ([5]). *If $E \in \mathfrak{M}_4$ and $x \in E$, then $\underline{d}(E, x) = \liminf\limits_{h \to 0^+} \frac{\lambda(E \cap [x-h, x+h])}{2h} > 0$.*

In view of the preceding sections it is also worth noting:

Proposition 6.1. *Each function f from the class $\mathcal{M}_i$, for $i \in \{0, 1, \dots, 6\}$, is almost continuous.*

In the case of discrete dynamical systems, functions whose domain and range is the same compact space are predominantly studied. In our considerations we will deal with the unit interval. So, from now on, we will focus on functions from the unit interval into itself. Let $i \in \{0, 1, \dots, 6\}$. The symbol $\mathcal{M}_i^{[0,1]}$ denote the family of all functions $f : [0,1] \circlearrowleft$ such that $f \in \mathcal{M}_i$.

Theorem 6.2. *If $f \in \mathcal{M}_i^{[0,1]}$ ($i \in \{0, 1, \dots, 6\}$), then* $\mathrm{Fix}(f) \neq \emptyset$.

Taking into account Proposition 6.1 and Theorem 5.2 we can immediately show that each function from $\mathcal{M}_i^{[0,1]}$ is Γ-approximated by continuous functions $\xi : [0,1] \circlearrowleft$. Furthermore, one can prove the following theorem, useful in various considerations.

Theorem 6.3. *If $f \in \mathcal{M}_i^{[0,1]}$ ($i \in \{0, 1, \dots, 6\}$), then f is Γ-approximated by continuous functions $\xi : [0,1] \circlearrowleft$ such that* $\mathrm{Fix}(\xi) \cap (0,1) \neq \emptyset$.

7. A topological entropy of discontinuous function from the unit interval into itself

As it has been already mentioned in the introduction, the problems connected with topological properties of dynamical systems (including topics related to topological entropy) can be also considered in the case of discontinuous functions.

In order to unify considerations we limit our further considerations only to the functions from the class $\mathcal{M}_1^{[0,1]}$ and finner classes of functions. **From now on we will assume that all functions belong to the class $\mathcal{M}_1^{[0,1]}$.** Now, we present some results related to issue connected with topological entropy of such functions.

Theorem 7.1 ([22]). *Let f be a function and $n \in \mathbb{N} \setminus \{1\}$. If $\{I_1, \dots, I_n\}$ is a family of pairwise disjoint closed intervals, then*

$$h(f) \geqslant \log \sigma(M_f(I_1, \dots, I_n)).$$

Theorem 7.2 ([22]). *A topological entropy of a function f equals 0 if and only if $h(f^n) = 0$ for each $n \in \mathbb{N}$.*

Theorem 7.3 ([22]). *If f is a turbulent function*[3] *then $h(f) > 0$.*

Theorem 7.4 (Itinerary Lemma, [28]). *For every function f and any family $\{I_1, \dots, I_n\}$ of closed intervals such that $I_1 \underset{f}{\rightarrow} I_2 \underset{f}{\rightarrow} \dots \underset{f}{\rightarrow} I_n \underset{f}{\rightarrow} I_1$ there exists $x_0 \in I_1$ such that $x_0 \in \mathrm{Fix}(f^n)$ and $f^i(x_0) \in I_{i+1}$ for $i \in \{1, \dots, n-1\}$.*

[3] A function f is turbulent if there exist compact subintervals $J, K \subset [0,1]$ with at most one common point such that $J \cup K \subset f(J) \cap f(K)$.

Theorem 7.5 (Sharkovskiĭ's Theorem, [28]). *Let $<_s$ be the linear ordering of $\mathbb{N}$ given by:*

$$3 <_s 5 <_s 7 < \ldots <_s 2\cdot 3 <_s 2\cdot 5 <_s \ldots <_s 2^2\cdot 3 <_s 2^2\cdot 5 <_s \ldots <_s 2^3 <_s 2^2 <_s 2 <_s 1$$

and $m, n \in \mathbb{N}$ be such that $n <_s m$. If a function f has a periodic orbit of period n, then f also has a periodic orbit of period m.

Theorem 7.6 ([7]). *A function f has a positive topological entropy if and only if f has a periodic point of period different from power of 2.*

As it has been already mentioned in the introduction, one can consider entropy connected with f-invariant measures. In [7] it was investigated in connections to Darboux-like functions. Now, we shortly recall this concept. Let $(X, \mathcal{S}, \mu)$ be a probability measure space (i.e. $\mu(X) = 1$), $f : X \circlearrowleft$ be a μ-measurable function and μ be an f-invariant measure (i.e. $\mu(f^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{S}$). Let $\mathcal{F} = \{A_i : i = 1, \ldots, m\}$ be a decomposition of X such that $A_i \in \mathcal{S}$ for $i \in \{1, \ldots, m\}$. If $R_{n-1}(\mathcal{F})$ is the set containing all intersections of the form $A_{i_1} \cap f^{-1}(A_{i_2}) \cap \cdots \cap f^{-(n-1)}(A_{i_n})$, then

$$h_\mu(f, \mathcal{F}) = -\lim_{n\to\infty} \frac{1}{n} \sum_{B \in R_{n-1}(\mathcal{F})} \mu(B) \cdot \log \mu(B).$$

The metric entropy, with respect to the measure μ, is the number

$$h_\mu(f) = \sup\{h_\mu(f, \mathcal{F}) : \mathcal{F} \subset \mathcal{S} \text{ is a finite decomposition of } X\}.$$

Theorem 7.7 ([7]). *For any function f we have that*

$$h(f) = \sup\{h_\mu(f) : \mu \textit{ is a probability } f\textit{-invariant Borel measure on } X\}.$$

Theorem 7.8 ([7]). *If a function f has a periodic point of period $2^k q$, where q is an odd number greater than 1 and $k \in \mathbb{N}$, then $h(f) > 0$.*

It is worth noting that the above statements are true for wider classes of functions ([7, 28]), but it should be mentioned here that in particular cases it may be more complex (see [19] in the context of the last theorem).

8. Almost fixed points of functions from the unit interval into itself belonging to the class $\mathcal{M}_1$

For simplicity of notation and considerations we will still discuss only functions belonging to $\mathcal{M}_1^{[0,1]}$. Definition of Darboux point ([6, 16]) and observations connected with entropy of Darboux-like functions led to distinguishing almost fixed points (see [22]). In general, almost fixed points have not been combined with strong entropy points. However, it occurs (see Corollary 8.4) that, under additional assumptions, these two notions can be combined. This situation is particularly interesting, because it regards only the case of discontinuous functions (an almost fixed point can not be a continuity point), thus the study of this notion is the original action distinguishing earlier research related to continuous functions.

In the case of considerations connected with the dynamical systems theory, particularly interesting are the properties being identical for conjugate functions. The following theorem shows that, from this point of view, the property of "being an almost fixed point" is important (of course within the scope of discontinuous Darboux functions).

Theorem 8.1 ([22]). *If functions f and g are topologically conjugate via a homeomorphism $\phi : [0,1] \circlearrowleft$ and $x_0 \in \mathrm{Fix}_a(f)$ then $\phi(x_0) \in \mathrm{Fix}_a(g)$.*

The next theorem will show that the name "almost fixed point" is not accidental.

Theorem 8.2 ([22]). *If f is a function with $x_0 \in \mathrm{Fix}_a(f)$ then $(x_0 - \delta, x_0 + \delta) \cap \mathrm{Fix}(f) \neq \emptyset$ for each $\delta > 0$.*

Theorem 8.3 ([22]). *If $\mathrm{Fix}_a(f)$ is nonempty for a function f, then $h(f) = \infty$.*

Let us introduce a notation $\mathrm{Fix}^*(f) = \mathrm{Fix}(f) \cap \mathrm{Fix}_a(f)$.

Corollary 8.4. *If f is a function such that $x_0 \in \mathrm{Fix}^*(f)$, then x_0 is a strong entropy point of f.*

Theorem 8.5 ([22]). *If f is a function with $\mathrm{Fix}_a(f) \neq \emptyset$, then $\mathrm{Per}_n(f) \neq \emptyset$ for any $n \in \mathbb{N}$.*

9. Approximation by functions from the Zahorski classes

Theorem 5.2 leads us to the following question: Is it possible to Γ-approximate functions from i-th Zahorski class ($i = 1, 2, 3, 4$) by use of functions from the same Zahorski class, having a strong entropy point and not belonging to $i + 1$ Zahorski class. Notice that the last requirement excludes continuous functions from the set of Γ-approximating functions (which leads to considerations essentially different from those regarding entropy of continuous functions). However, we can extend the question requiring Γ-approximating functions having only one discontinuity point, which moreover is a strong entropy point of this function.

First notice that in the case of $i = 1$ the last demand is impossible. Indeed, let $f \in \mathcal{M}_1^{[0,1]}$, $U \subset [0,1] \times [0,1]$ be a nonempty open set containing the graph of f and g be a function from $\mathcal{M}_1^{[0,1]} \setminus \mathcal{M}_2^{[0,1]}$ whose graph belongs to U. Without loss of generality we can assume that there exist $a \in \mathbb{R}$ and a set $E^a = \{x : g(x) > a\}$ such that $E^a \in \mathfrak{M}_1 \setminus \mathfrak{M}_2$. There is no loss of generality in assuming that there are a point $x \in E^a$ and $h > 0$ such that $\lambda((x, x+h) \cap E^a) = 0$. Obviously $(x, x+h) \cap E^a \neq \emptyset$ and each point from $(x, x+h) \cap E^a$ is a discontinuity point of g.

However, the following theorem holds:

Theorem 9.1 ([13]). *Let $i \in \{1, 2, \ldots, 5\}$. Each function from the class $\mathcal{M}_i^{[0,1]}$ can be Γ-approximated by functions belonging to the class $\mathcal{M}_i^{[0,1]} \setminus \mathcal{M}_{i+1}^{[0,1]}$ and having a strong entropy point. Moreover, if $i \neq 1$ then each function from the class $\mathcal{M}_i^{[0,1]}$ can be Γ-approximated by functions belonging to the class $\mathcal{M}_i^{[0,1]} \setminus \mathcal{M}_{i+1}^{[0,1]}$ and having a strong entropy point which is simultaneously the only one discontinuity point.*

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